

AN ALTERNATIVE PROOF OF KAZHDAN PROPERTY FOR ELEMENTARY GROUPS

MASATO MIMURA

ABSTRACT. In 2010, Invent. Math., Ershov and Jaikin–Zapirain proved Kazhdan’s property (T) for elementary groups. This expository article focuses on presenting an alternative simpler proof. Unlike the original one, our proof supplies *no* estimate of Kazhdan constants. It is a specific example of the paper “*Super-intrinsic synthesis in fixed point properties*” (arXiv:1505.06728) by the author.

1. INTRODUCTION

Throughout this article, \mathcal{H} stands for an arbitrary Hilbert space (we do not fix a single Hilbert space: it can be non-separable after taking metric ultraproducts).

Definition 1.1. For a countable group G and $G \geq M$, we say that $G \geq M$ has *relative property* (FH) if for all (affine) isometric G -actions $\alpha: G \curvearrowright \mathcal{H}$ (for every \mathcal{H}), the M -fixed point set $\mathcal{H}^{\alpha(M)}$ is non-empty. We say that G has *property* (FH) if $G \geq G$ has relative property (FH).

The Delorme–Guichardet Theorem [BdlHV08, Theorem 2.12.4] states that (relative) property (FH) is equivalent to (relative) *property* (T) of Kazhdan. Therefore, throughout this article, we use the terminology “property (T)” for property (FH). See [BdlHV08] for details on these properties. A fundamental example of groups with property (T) is $\mathrm{SL}(n, \mathbb{Z})$ for $n \geq 3$ (see [BdlHV08, Example 1.7.4.(i)]).

The goal of this article is to provide an alternative proof of the following theorem:

Theorem 1.2 (Ershov and Jaikin–Zapirain, Theorem 1 in [EJZ10]). *For a finitely generated and associative ring R with unit and for $n \geq 3$, the elementary group $E(n, R)$ has property (T).*

Here, for such R and n , the *elementary group* $E(n, R)$ is the subgroup of $\mathrm{GL}(n, R)$ generated by *elementary matrices* $\{e_{i,j}(r) : i \neq j \in \{1, 2, \dots, n\}, r \in R\}$. The $e_{i,j}(r)$ is defined by $(e_{i,j}(r))_{k,l} := \delta_{k,l} + r\delta_{i,k}\delta_{j,l}$, where $\delta_{\cdot,\cdot}$ is the Dirac delta. This theorem greatly generalizes the aforementioned example because for $R = \mathbb{Z}$, $\mathrm{SL} = E$ (Gaussian elimination). The commutator relation

$$[e_{i,j}(r_1), e_{j,k}(r_2)] := e_{i,j}(r_1)e_{j,k}(r_2)e_{i,j}(r_1)^{-1}e_{j,k}(r_2)^{-1} = e_{i,k}(r_1r_2) \quad \cdots (*)$$

for $i \neq j \neq k \neq i$ implies finite generation of $E(n, R)$ as in Theorem 1.2.

Note: in [EJZ10], $E(n, R)$ is written as $EL_n(R)$. The ring R may be non-commutative.

For motivations of this result, see Introduction of [EJZ10].

Date: November 2, 2016.

Key words and phrases. Kazhdan’s property (T); elementary groups.

2. STRATEGIES: COMMON POINTS AND DIFFERENCE

Both of the original proof, and the new proof in this article adopt the “*the Part and the Whole strategy*”: for $G = E(n, R)$,

- (“Part Step”): show relative properties (T) for $G \geq M_i$ for certain subgroups M_1, \dots, M_l .
- (“Synthesis Step”): synthesize them into property (T) for G .

In “Part Step”, both of us employ

Theorem 2.1 (Kassabov, Corollary 2.8 in [Kas07]). *For R and n as in Theorem 1.2, and for all $i \neq j \in \{1, \dots, n\}$, $E(n, R) \geq G_{i,j}$ has relative property (T). Here, $G_{i,j} := \langle e_{i,j}(r) : r \in R \rangle (\simeq (R, +))$.*

Note: Kassabov showed it in terms of the original definition of property (T). As we mentioned above, this is equivalent to our property (T) (property (FH)).

In fact, Kassabov’s original form is for the pair $E(n-1, R) \ltimes R^{n-1} \supseteq R^{n-1}$ for all $n \geq 3$, where $E(n-1, R)$ naturally acts on R^{n-1} . The proof is by spectral theory associated with unitary representations of abelian groups: see also [Sha99] and [BdlHV08, Sections 4.2 and 4.3]. To deduce Theorem 2.1 from this, embed $E(n-1, R) \ltimes R^{n-1}$ into $E(n, R)$ in several ways. Note that if $G \geq G_0 \geq M_0 \geq M$, then relative property (T) for $G \geq M$ follows from that for $G_0 \geq M_0$.

The difference between the original proof in [EJZ10] and ours is “Synthesis Step”.

- (1) In the original proof, synthesis is *extrinsic*: They consider *angles* between fixed point subspaces, and showed that if angles are sufficiently close to being orthogonal, then synthesis holds.

More precise form is as follows. Since their original argument deals with the original formulation of property (T), we here rather sketch the argument in formulation of Lavy [Lav15]. For two (non-empty) affine subspaces $\mathcal{K}_1, \mathcal{K}_2$ of \mathcal{H} , they define

$$\cos \angle(\mathcal{K}_1, \mathcal{K}_2) := \sup_{0 \neq \xi \in \mathcal{K}'_1 / (\mathcal{K}'_1 \cap \mathcal{K}'_2), 0 \neq \eta \in \mathcal{K}'_2 / (\mathcal{K}'_1 \cap \mathcal{K}'_2)} \frac{|\langle \xi, \eta \rangle|}{\|\xi\| \|\eta\|} \in [0, 1],$$

where $\mathcal{K}'_i, i = 1, 2$, is the *linear* subspace obtained by parallel transformation of \mathcal{K}_i . The symbol $\langle \cdot, \cdot \rangle$ means the (induced) inner product on $\mathcal{H} / (\mathcal{K}'_1 \cap \mathcal{K}'_2)$. They say \mathcal{K}_1 and \mathcal{K}_2 are ϵ -orthogonal if $\cos \angle(\mathcal{K}_1, \mathcal{K}_2) < \epsilon$.

They showed existence of (explicit) $(\epsilon_{i,j})_{1 \leq i < j \leq l}$ with the following property: for $G \geq H_i, 1 \leq i \leq l$, with $\langle H_1, \dots, H_l \rangle = G$, if $\mathcal{H}^{\alpha(H_i)}$ and $\mathcal{H}^{\alpha(H_j)}$ are $\epsilon_{i,j}$ -orthogonal for all $i \neq j$ and for all affine isometric action $\alpha: G \curvearrowright \mathcal{H}$, then “relative properties (T) for $G \geq H_{i,j} := \langle H_i, H_j \rangle$ for all $i \neq j$ ” imply property (T) for G . See [Lav15, Subsection 1.2 and Section 2] as well as [EJZ10].

Note: to apply this “Synthesis” criterion for $G = E(n, R)$ for a general R (say, $R = \mathbb{Z}\langle x, y \rangle$, a non-commutative polynomial ring), quite delicate estimate of spectral quantities ($\epsilon_{i,j}$ for $\epsilon_{i,j}$ -orthogonality) is needed. This, together with the proof of the criterion, makes their proof heavier. In return, they obtain estimation of *Kazhdan constants* (see [BdlHV08, Remark 1.1.4]).

- (2) Our synthesis is *intrinsic*: our “Synthesis” criterion is stated only in terms of group structure, and not of group actions. This lets our proof rather simpler than the original one. The price to pay, however, is that we do *not* obtain any estimate for Kazhdan constants.

This is a special case of our *superintrinsic synthesis* [Mim15, Theorem B].

Theorem 2.2 (Our Synthesis Step). *Let G be a finitely generated group and $M, L \leq G$. Assume the following two hypotheses:*

- (i) $\langle M, L \rangle = G$; and
 (GAME⁺) *the player can win the (Game⁺) for (M, L) , which is defined in Section 3.*
Then, relative properties for $G \geq M$ and $G \geq L$ imply property (T) for G .

3. THE (Game⁺)

The “(Game⁺) for (M, L) ” is a one-player game. Here, we fix (M, L) , where $M, L \leq G$, and keep them unchanged. We set $H_1 \leq G$ and $H_2 \leq G$: those two vary stage by stage in the game. The rules are:

- in the initial stage, $H_1 = M$ and $H_2 = L$;
- in each stage, the player is allowed to enlarge both of H_1 and H_2 by admissible moves (I⁺) and (II’), which we define below; and
- the winning condition is that within finite steps of moves, the player sets either $H_1 = G$ or $H_2 = G$.

(Rules of the moves.)

- *Type (I⁺) move*: pick a subset $P \subseteq G$ such that for all $h \in P$,

$$hH_1h^{-1} \geq M \quad \text{and} \quad hH_2h^{-1} \geq L.$$

Then enlarge H_1 and H_2 as:

$$\frac{H_1}{\langle H_1, P \rangle} \quad \bigg| \quad \frac{H_2}{\langle H_2, P \rangle}.$$

Here, this table means that H_1 is enlarged to $\langle H_1, P \rangle$, and this $\langle H_1, P \rangle$ is set as new H_1 after this move; and similarly on H_2 . We use similar tables to indicate this kind of enlargements.

- *Type (II’) move*: pick a subset $W \subseteq G$ such that for all $w \in W$,

$$wH_2w^{-1} \geq M \quad \text{and} \quad wH_1w^{-1} \geq L.$$

Then enlarge H_1 and H_2 as:

$$\frac{H_1}{\langle H_1, \bigcup_{w \in W} w^{-1}H_2w \rangle} \quad \bigg| \quad \frac{H_2}{\langle H_2, \bigcup_{w \in W} w^{-1}H_1w \rangle}.$$

Here, (Game⁺) and (I⁺) are *relaxed* versions of (Game) and (I). The ‘ means that (II’) is a *restricted* case of (II). Compare with [Mim15, Theorem A and Theorem B].

This (Game⁺) may look frightening, but it is in fact natural: it represents ways of *self-improvement*. Rough meaning is: for an affine isometric action $\alpha: G \curvearrowright \mathcal{H}$,

“if $\xi \in \mathcal{H}^{\alpha(M)}$ and $\eta \in \mathcal{H}^{\alpha(M)}$ are chosen in a special manner, then *automatically* $\xi \in \mathcal{H}^{\alpha(H_1)}$ and $\eta \in \mathcal{H}^{\alpha(H_2)}$ in each stage of $(\text{Game}^{+'})$.” See Proposition 4.1 for the rigorous statement. Note that, although precise forms are rather different, some sort of self-improvement is well-known as *Mautner phenomena* in continuous setting (unitary representations of Lie groups): see [BdlHV08, Lemma 1.4.8].

Let us see how we employ this criterion to the case of $G = E(n, R)$.

Proof of “Theorem 2.2 implies Theorem 1.2”. Let R and n be as in Theorem 1.2. Set $G = E(n, R)$, $M = \begin{pmatrix} I_{n-1} & R^{n-1} \\ 0 & 1 \end{pmatrix} (\simeq (R^{n-1}, +))$, and $L = \begin{pmatrix} I_{n-1} & 0 \\ {}^t(R^{n-1}) & 1 \end{pmatrix}$.

Lemma 3.1. *These $G \geq M$ and $G \geq L$ have relative property (T).*

Proof. This follows from Theorem 2.1. Indeed, $L = G_{n,1} \cdot G_{n,2} \cdots G_{n,n-1}$. Hence it follows that for every affine isometric action $\alpha: G \curvearrowright \mathcal{H}$, some (equivalently, all) $\alpha(L)$ -orbit is bounded (observe that each $\alpha(G_{i,j})$ -orbits are bounded by Theorem 2.1). Then, the *Chebyshev center* of an $\alpha(L)$ -orbit ([BdlHV08, Definition 2.2.8]) is $\alpha(L)$ -fixed. The case for M is immediate by original form of Kassabov. \square

It remains to check our hypotheses (i) and $(\text{GAME}^{+'})$: (i) follows from (*). For $(\text{GAME}^{+'})$, first, take type (I^+) move for $P = \begin{pmatrix} E(n-1, R) & 0 \\ 0 & 1 \end{pmatrix}$. This enlarges $H_1 = M$ and $H_2 = L$ as

$$\frac{H_1}{\langle H_1, P \rangle = \begin{pmatrix} E(n-1, R) & R^{n-1} \\ 0 & 1 \end{pmatrix}} \quad \Bigg| \quad \frac{H_2}{\langle H_2, P \rangle = \begin{pmatrix} E(n-1, R) & 0 \\ {}^t(R^{n-1}) & 1 \end{pmatrix}}.$$

Finally, take type (II') move with $W = \{w\}$. Here, $w = \begin{pmatrix} 0 & 0 & 1 \\ 0 & I_{n-2} & 0 \\ -1 & 0 & 0 \end{pmatrix}$. Note that $w = e_{1,n}(1)e_{n,1}(-1)e_{1,n}(1) \in G$. The *key* here is $wH_2w^{-1} = \begin{pmatrix} 1 & {}^t(R^{n-1}) \\ 0 & E(n-1, R) \end{pmatrix} \geq M$, and $wH_1w^{-1} = \begin{pmatrix} 1 & 0 \\ R^{n-1} & E(n-1, R) \end{pmatrix} \geq L$. Therefore, H_1 and H_2 are enlarged as

$$\frac{H_1}{\langle H_1, w^{-1}H_2w \rangle = \left\langle \begin{pmatrix} E(n-1, R) & R^{n-1} \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & {}^t(R^{n-1}) \\ 0 & E(n-1, R) \end{pmatrix} \right\rangle} \quad \Bigg| \quad \frac{H_2}{\langle H_2, w^{-1}H_1w \rangle}.$$

The new H_1 and H_2 both equal G by (*). We are done. \square

We will prove Theorem 2.2 in Section 4: the combination of these proofs provides our new proof of Theorem 1.2.

Remark 3.2. We call such synthesis *superintrinsic synthesis* [Mim15]. Here, “*super*” means we do not impose “Bounded Generation” condition on the hypotheses.

Definition 3.3. A subset $(1_G \in)U$ of G are said to *Boundedly Generate* G if there exists $N \in \mathbb{N}$ such that every $g \in G$ may be written as the product of N (possibly overlapping) elements in U .

Note: in some literature, “Bounded Generation” is used in the following very restrictive way: $U = \bigcup_{1 \leq i \leq l} C_i$ for C_i cyclic. Our convention is much more general.

Study of intrinsic syntheses was initiated and developed by works of Shalom [Sha99], [Sha06]. There, Bounded Generation hypotheses were essential: see also [BdlHV08, Chapter 4] for the former work. This *was* a bottle-neck for intrinsic syntheses, because it is very strong in general. For instance, Bounded Generation for $E(n, R)$, $n \geq 3$, by elementary matrices $(\bigcup_{i \neq j} G_{i,j})$, where $G_{i,j}$ is as in Theorem 2.1) is true for $R = \mathbb{Z}$ (Carter–Keller, see [BdlHV08, Section 4.1]); *false* for $R = \mathbb{C}[x]$ for every n (van der Kallen); and *open* in many cases, even for $R = \mathbb{Z}[x]$.

Our Theorem 2.2 is *superintrinsic*, and applies to general $E(n, R)$, as above.

4. PROOF OF THEOREM 2.2

4.1. Self-improvement argument. The idea of the proof is inspired by Shalom’s second intrinsic synthesis [Sha06, 4.III]: Shalom himself called it *algebraization*. Our *self-improvement argument* is as follows:

Proposition 4.1 (Key proposition: *Self-improvement*). *Let G be a finitely generated group. Let $M, L \leq G$ with $\langle M, L \rangle = G$. Let $\alpha: G \curvearrowright \mathcal{H}$ be an affine isometric action. Assume that the linear part π of α does not have non-zero G -invariant vectors. Assume $\mathcal{H}^{\alpha(M)} \neq \emptyset$ and $\mathcal{H}^{\alpha(L)} \neq \emptyset$. Assume, besides, that (ξ, η) , where $\xi \in \mathcal{H}^{\alpha(M)}$ and $\eta \in \mathcal{H}^{\alpha(L)}$, realizes the distance*

$$D := \text{dist}(\mathcal{H}^{\alpha(M)}, \mathcal{H}^{\alpha(L)}).$$

Then, in each stage in (Game^{+}) for (M, L) , $\xi \in \mathcal{H}^{\alpha(H_1)}$ and $\eta \in \mathcal{H}^{\alpha(H_2)}$.

Recall an affine isometric action α is decomposed into linear part π (unitary representation) and cocycle part b ; $\alpha(\gamma) \cdot \zeta = \pi(\gamma)\zeta + b(\gamma)$ (see [BdlHV08, Section 2.1]).

The key to the proof is the following simple observation due to Shalom.

Lemma 4.2 (*Shalom’s parallelogram argument*, see 4.III.6 in [Sha06]). *In the setting of Proposition 4.1, the realizer of D is unique.*

Proof. Let (ξ', η') be another realizer. Take midpoints (m_1, m_2) , where $m_1 = (\xi + \xi')/2$ and $m_2 = (\eta + \eta')/2$. Observe that (m_1, m_2) is again a realizer of D , because $m_1 \in \mathcal{H}^{\alpha(M)}$, $m_2 \in \mathcal{H}^{\alpha(L)}$, and $\|m_1 - m_2\| \leq D$ (by triangle inequality).

Then, by strict convexity of \mathcal{H} , $\xi - \eta = \xi' - \eta'$. Note $\xi - \xi' \in \mathcal{H}^{\pi(M)}$ and $\eta - \eta' \in \mathcal{H}^{\pi(L)}$. Therefore, $\xi - \xi' = \eta - \eta' \in \mathcal{H}^{\pi(G)} (= \{0\})$, as claimed. \square

Proof of Proposition 4.1. By induction on number of moves. In the initial stage, the assertion holds. We proceed in induction step: assume that $\xi \in \mathcal{H}^{\alpha(H_1)}$ and $\eta \in \mathcal{H}^{\alpha(H_2)}$ for the current H_1 and H_2 ; and we take a new move.

- *Case 1. New move is of type (I^+) :* this case was essentially done by Shalom [Sha06, 4.III.6]. Let $h \in P$. The conditions $hH_1h^{-1} \geq M$ and $hH_2h^{-1} \geq L$ are imposed on P exactly to ensure $\alpha(h) \cdot \xi \in \mathcal{H}^{\alpha(M)}$ and $\alpha(h) \cdot \eta \in \mathcal{H}^{\alpha(L)}$. By

isometry of α , (ξ, η) and $(\alpha(h) \cdot \xi, \alpha(h) \cdot \eta)$ are two realizers of D . Therefore, Lemma 4.2 applies. We obtain that $\xi \in \mathcal{H}^{\alpha(P)} \cap \mathcal{H}^{\alpha(H_1)} = \mathcal{H}^{\alpha(\langle H_1, P \rangle)}$. A similar argument verifies the case for η .

- *Case 2. New move is of type (Π') :* Let $w \in W$. Then, the condition on w implies $\alpha(w) \cdot \eta \in \mathcal{H}^{\alpha(M)}$ and $\alpha(w) \cdot \xi \in \mathcal{H}^{\alpha(L)}$. Hence, this time $(\alpha(w) \cdot \eta, \alpha(w) \cdot \xi)$ is another realizer of D . Again by Lemma 4.2, $\alpha(w) \cdot \xi = \eta$. By recalling $\eta \in \mathcal{H}^{\alpha(H_2)}$, we conclude that $\xi \in \mathcal{H}^{\alpha(w^{-1}H_2w)}$ for all $w \in W$. A similar argument applies to η .

□

4.2. Metric ultraproducts, scaling limits, and realizers of the distance.

Proposition 4.1 might look convincing, but there is a *gap* to conclude Theorem 2.2: in general, there is *no* guarantee of the existence of *realizers* (ξ, η) of D .

This gap will be fixed by well-known Propositions 4.4 and 4.5 below. Nevertheless, we include (sketchy) proofs for the reader's convenience. They employ metric ultraproducts. The reader who is familiar with this topic may skip this subsection.

Definition 4.3 (Displacement and uniform action). Fix a finite generating set S of a finitely generated group G . Let $\alpha: G \curvearrowright \mathcal{H}$ be an action.

- (1) The *displacement function* is defined by

$$\text{disp}_{\alpha, S} =: \text{disp}_\alpha: \mathcal{H} \rightarrow \mathbb{R}_{\geq 0}; \quad \zeta \mapsto \max_{s \in S} \|\alpha(s) \cdot \zeta - \zeta\|.$$

- (2) The action α is said to be *1-uniform* if $\inf_{\zeta \in \mathcal{H}} \text{disp}_\alpha(\zeta) \geq 1$.

Fix a finitely generated group G and a finite generating set S . We set the following three classes of actions and Hilbert spaces:

- $\mathcal{C} := \{(\alpha, \mathcal{H})\}$, where \mathcal{H} is a Hilbert space and $\alpha: G \curvearrowright \mathcal{H}$ is an affine isometric action;
- $\mathcal{C}^{\text{non-fixed}} := \{(\alpha, \mathcal{H}) : (\alpha, \mathcal{H}) \in \mathcal{C}, \mathcal{H}^{\alpha(G)} = \emptyset\}$; and
- $\mathcal{C}^{1\text{-uniform}} := \{(\alpha, \mathcal{H}) : (\alpha, \mathcal{H}) \in \mathcal{C}, \alpha \text{ is 1-uniform}\}.$

Then, $\mathcal{C}^{1\text{-uniform}}$ is a subclass of $\mathcal{C}^{\text{non-fixed}}$. The failure of property (T) for G exactly says that $\mathcal{C}^{\text{non-fixed}} \neq \emptyset$.

In the two propositions below, let G be a finitely generated group, and fix a finite generating set S .

Proposition 4.4 (A special case of the *Gromov–Schoen argument*, see also 4.III.2 in [Sha06]). *Assume G fails to have property (T). Then, $\mathcal{C}^{1\text{-uniform}} \neq \emptyset$.*

Proposition 4.5 (Shalom, 4.III.3–4 in [Sha06]). *Let $M \leq G$ and $L \leq G$ with $\langle M, L \rangle = G$. Assume that $M \leq G$ and $L \leq G$ have relative property (T); and that $\mathcal{C}^{1\text{-uniform}} \neq \emptyset$.*

Then, $D := \inf\{\|\xi - \eta\| : (\alpha, \mathcal{H}) \in \mathcal{C}^{1\text{-uniform}}, \xi \in \mathcal{H}^{\alpha(M)}, \eta \in \mathcal{H}^{\alpha(L)}\}$ is realized.

Here we briefly recall the definitions on (pointed) metric ultraproducts. See a survey [Sta09] for more details. *Ultrafilters* \mathcal{U} on \mathbb{N} have one-to-one correspondence to $\{0, 1\}$ -valued probability *means* (that means, *finitely additive* measures μ defined over all subsets in \mathbb{N} with $\mu(\mathbb{N}) = 1$). The correspondence is in the following manner:

$\mathcal{U} = \{A \subseteq \mathbb{N} : \mu(A) = 1\}$. A *principal* ultrafilter corresponds to the Dirac mass at a point in \mathbb{N} . *Non-principal* ultrafilters correspond to all the other ones.

In what follows, we fix a non-principal ultrafilter \mathcal{U} . For real numbers r_n , we write as $\lim_{\mathcal{U}} r_n = r_{\infty}$, if for all $\epsilon > 0$, $\{n \in \mathbb{N} : |r_{\infty} - r_n| < \epsilon\} \in \mathcal{U}$. Then, it is well-known that every *bounded* real sequence $(r_n)_n$ has a (unique) limit with respect to \mathcal{U} . Since \mathcal{U} is non-principal, if $\lim_{n \rightarrow \infty} r_n$ exists, then it coincides with $\lim_{\mathcal{U}} r_n$.

Let $((X_n, d_n, z_n))_{n \in \mathbb{N}}$ be a sequence of pointed metric spaces. Let $\ell_{\infty}\text{-}\sum(X_n, z_n) := \{(x_n)_n : x_n \in X_n, \sup_{n \in \mathbb{N}} d_n(x_n, z_n) < \infty\}$, and $d_{\infty}((x_n)_n, (y_n)_n) := \lim_{\mathcal{U}} d_n(x_n, y_n)$. Finally, define the *pointed metric ultraproduct* $(X_{\mathcal{U}}, d_{\mathcal{U}}, z_{\mathcal{U}})$ as follows:

$$X_{\mathcal{U}} := (\ell_{\infty}\text{-}\sum(X_n, z_n)) / \sim_{d_{\infty}=0},$$

$d_{\mathcal{U}}$ is the induced (genuine) metric, and $z_{\mathcal{U}} := [(z_n)_n]$, where $[\cdot]$ is the equivalence class. This is also written as $\lim_{\mathcal{U}}(X_n, d_n, z_n)$. We can show that metric ultraproducts of (affine) Hilbert spaces are again (affine) Hilbert spaces (because Hilbert spaces are characterized in terms of inner products).

We fix (G, S) . For a sequence of pointed (isometric) G -actions $(\alpha_n, (X_n, d_n), z_n)$ that satisfies

$$\sup_n \text{disp}_{\alpha_n}(z_n) < \infty \quad \cdots (\diamond),$$

we can define the *pointed metric ultraproduct action* $\alpha_{\mathcal{U}}$ on $(X_{\mathcal{U}}, d_{\mathcal{U}}, z_{\mathcal{U}})$ by $\alpha_{\mathcal{U}}(\gamma) \cdot [(x_n)_n] := [(\alpha_n(\gamma) \cdot x_n)_n]$. This is also written as $\lim_{\mathcal{U}}(\alpha_n, X_n, z_n)$.

Proof of Proposition 4.4. Let $(\alpha, \mathcal{H}) \in \mathcal{C}^{\text{non-fixed}} (\neq \emptyset)$. By employing completeness of \mathcal{H} , we can find a sequence $(\zeta_n)_{n \in \mathbb{N}}$ with the following property: for all $\chi \in \mathcal{H}$ with $\|\chi - \zeta_n\| \leq (n+1)\text{disp}_{\alpha}(\zeta_n)$, $\text{disp}_{\alpha}(\chi) \geq \text{disp}_{\alpha}(\zeta_n)/2$. For details, see [Sta09, Lemma 3.3].

Then, the ultraproduct $\lim_{\mathcal{U}}(\alpha, (\mathcal{H}, r_n \|\cdot\|), \zeta_n)$ is well-defined and 1-uniform. Here $r_n := 2(\text{disp}_{\alpha}(\zeta_n))^{-1}$. \square

Proof of Proposition 4.5. Observe that this infimum is over a non-empty set. Let $((\alpha_n, \mathcal{H}_n, \xi_n, \eta_n))_n$ be a sequence that “asymptotically realizes” D as $n \rightarrow \infty$. More precisely, assume $\|\xi_n - \eta_n\| \leq D + 2^{-n}$. We claim that $((\alpha_n, \mathcal{H}_n, \xi_n))_{n \in \mathbb{N}}$ satisfies (\diamond) . Indeed, note that $\sup_{\gamma \in M} \|\alpha_n(\gamma) \cdot \xi_n - \xi_n\| = 0$ and $\sup_{\gamma \in L} \|\alpha_n(\gamma) \cdot \eta_n - \eta_n\| = 0$. Then, by isometry of α_n and triangle inequality, $\sup_{\gamma \in M \cup L} \|\alpha_n(\gamma) \cdot \xi_n - \xi_n\| \leq 2(D + 2^{-n})$. Then, because $\|\alpha_n(gh) \cdot \zeta - \zeta\| \leq \|\alpha_n(h) \cdot \zeta - \zeta\| + \|\alpha_n(g) \cdot \zeta - \zeta\|$ in general,

$$\sup_n \text{disp}_{\alpha_n}(\xi_n) \leq \sup_n 2N(D + 2^{-n}) \leq 2N(D + 1) < \infty.$$

Here $N(< \infty)$ is the maximum of word length on S with respect to $M \cup L$.

Finally, the resulting action $(\alpha, \mathcal{H}) := \lim_{\mathcal{U}}(\alpha_n, \mathcal{H}_n)$ (with base points $(\xi_n)_n$); and $\xi := [(\xi_n)_n]$ and $\eta := [(\eta_n)_n]$ realize D . Here, observe that $(\alpha, \mathcal{H}) \in \mathcal{C}^{1\text{-uniform}}$. \square

4.3. Closing.

Proof of Theorem 2.2. By contradiction. Suppose that $G \geq M$ and $G \geq L$ have relative property (T); but that G fails to have property (T). Then, by Propositions 4.4 and 4.5, there must exist a realizer $(\alpha, \mathcal{H}, \xi, \eta)$ of D as in Proposition 4.5. In particular, $\|\xi - \eta\| = D = \text{dist}(\mathcal{H}^{\alpha(M)}, \mathcal{H}^{\alpha(L)})$.

Observe $G^{\text{abel}} := G/[G, G]$ is finite. Indeed, for the abelianization map $\text{ab}_G: G \twoheadrightarrow (G^{\text{abel}}, +)$, relative properties (T) above imply $|\text{ab}_G(M)| < \infty$ and $|\text{ab}_G(L)| < \infty$: otherwise, we would have non-trivial translations (finite generation of G implies ones of $\text{ab}_G(M)$ and $\text{ab}_G(L)$). Since $\text{ab}_G(G) = \text{ab}_G(M) + \text{ab}_G(L)$, we are done.

Let π be the linear part of α . According to the decomposition $\mathcal{H} = \mathcal{H}^{\pi(G)} \oplus (\mathcal{H}^{\pi(G)})^\perp$, α is decomposed into α_{trivial} and $\alpha_{\text{orthogonal}}$ (this is done by decomposing the cocycle b into these two summands). Because G^{abel} is finite, α_{trivial} is the trivial action. Therefore, we can extract $\alpha_{\text{orthogonal}}$ from α , without changing D and 1-uniformity. We, thus, may assume that $\mathcal{H}^{\pi(G)} = \{0\}$.

Then, Proposition 4.1 applies: either $\xi \in \mathcal{H}^{\alpha(G)}$ or $\eta \in \mathcal{H}^{\alpha(G)}$ must hold by hypothesis (GAME⁺). It, however, contradicts that α is 1-uniform. \square

Remark 4.6. Our Theorem 2.2 is greatly generalized to [Mim15, Theorem A]: we deal with fixed point properties relative to more general metric spaces (even non-linear ones); and we allow some non-inner automorphisms of G in type (II) moves.

ACKNOWLEDGMENTS

The author is truly indebted to Michihiko Fujii for the kind invitation to the conference “Topology and Analysis of Discrete Groups and Hyperbolic Spaces” in June, 2016 at the RIMS, Kyoto, where this expository article had an occasion to come out. He thanks Pierre de la Harpe for drawing the author’s attention to the Mautner phenomenon; Andrei Jaikin-Zapirain for comments; Masahiko Kanai for providing him with the terminology “intrinsic”; and Takayuki Okuda for discussions on moves in (Game). The author is supported in part by JSPS KAKENHI Grant Number JP25800033 and by the ERC grant 257110 “RaWG”.

REFERENCES

- [BdlHV08] Bachir Bekka, Pierre de la Harpe, and Alain Valette, *Kazhdan’s property (T)*, New Mathematical Monographs, vol. 11, Cambridge University Press, Cambridge, 2008. MR 2415834
- [EJZ10] Mikhail Ershov and Andrei Jaikin-Zapirain, *Property (T) for noncommutative universal lattices*, Invent. Math. **179** (2010), no. 2, 303–347. MR 2570119
- [Kas07] Martin Kassabov, *Universal lattices and unbounded rank expanders*, Invent. Math. **170** (2007), no. 2, 297–326. MR 2342638
- [Lav15] Omer Lavy, *Fixed point theorems for groups acting on non-positively curved manifolds*, preprint, arXiv: 1512.07745v2 (2015).
- [Mim15] Masato Mimura, *Superintrinsic synthesis in fixed point properties*, forthcoming version (v2) of the preprint on arXiv: 1505.06728 (2015).
- [Sha99] Yehuda Shalom, *Bounded generation and Kazhdan’s property (T)*, Inst. Hautes Études Sci. Publ. Math. (1999), no. 90, 145–168 (2001). MR 1813225
- [Sha06] ———, *The algebraization of Kazhdan’s property (T)*, International Congress of Mathematicians. Vol. II, Eur. Math. Soc., Zürich, 2006, pp. 1283–1310. MR 2275645
- [Sta09] Yves Stalder, *Fixed point properties in the space of marked groups*, Limits of graphs in group theory and computer science, EPFL Press, Lausanne, 2009, pp. 171–182. MR 2562144